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Normalized Ground State Solutions to Fractional Schrödinger Equations with General Nonlinearities

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Abstract

This paper investigates the existence of normalized solutions to a fractional Schrödinger equation with combined nonlinearities. In previous studies, the equation $-\Delta u+\lambda u=g(u)+|u|^{q-2}u$. where $N\geq 3, 2<q<2^*=\frac{2N}{N-2}$ has been proven to have solutions through various constraints and methods. Furthermore, we consider the existence of solutions for fractional equations. In the L^2-supercritical cases, we employ the Sobolev subcritical approximation method to establish the existence of normalized ground-state solutions.

Subject Areas

Functional Analysis

Keywords

Fractional Schrödinger Equation, Normalized Solution, Constrained Minimization

1. Introduction

In this paper, we are concerned with the existence of normalized solutions to the following Schrödinger equation with combined nonlinearities:

$$\begin{cases}(-\Delta)^s u+\lambda u=g(u)+|u|^{q-2}u, & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=a, & u\in H^s(\mathbb{R}^N),\end{cases} \tag{1.1}$$

where $\frac{1}{2}<s<1$, $N\geq 2$, $\lambda\in\mathbb{R}$, $2+\frac{4s}{N}<q<2_s^*=\frac{2N}{N-2s}$ and g satisfies the following conditions:

$(G1)$ $g \in C(\mathbb{R}, \mathbb{R})$ is an odd function and $G(u) = \int_0^u g(t)\,\mathrm{d}t$;

$(G2)$ $\limsup_{u \to 0} \frac{g(u)}{|u|^{1+\frac{4s}{N}}} = \beta \in [0, \infty)$ and $\lim_{u \to +\infty} \frac{g(u)}{|u|^{2_s^*-1}} = 0$;

$(G3)$ $\frac{\tilde{G}(u)}{|u|^{2+\frac{4s}{N}}}$ is increasing on $(0, +\infty)$, where $\tilde{G}(u) = g(u)u - 2G(u)$;

$(G4)$ $g(u)u < 2_s^* G(u)$, for all $u \in \mathbb{R} \setminus \{0\}$.

$(-\Delta)^s$ is the fractional Laplace operator defined as

$$(-\Delta)^s u(x) = C(N, s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

for $u \in C_0^\infty(\mathbb{R}^N)$, where $C(N, s)$ is a suitable positive constant and P.V. denotes the Cauchy principal value. We note that since the fractional Laplace operator has nonlocal properties, it leads to more challenges compared with the classical Laplace operator from a mathematical point of view. We refer the interested reader to [1]-[3] for a preliminary introduction to the fractional Laplace operator and fractional Sobolev spaces.

Our main driving force for the study of (1.1) arises in the study of the following time-dependent fractional Schrödinger equation:

$$\begin{cases} i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi - g(|\psi|)\psi, & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ \int_{\mathbb{R}^N} |\psi(x,t)|^2\,\mathrm{d}x = a, & \text{for all } t \in \mathbb{R}^+, \end{cases} \tag{1.2}$$

where $g(t) = t^{q-2} + t^{p-2}$, $2 < q < p \le 2_s^*$ and i stands for the imaginary unit. When searching for stationary waves of the form $\psi(x,t) = \mathrm{e}^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ is the chemical potential and $u(x): \mathbb{R}^N \to \mathbb{C}$ is a time-independent function in quantum mechanics, one is led to studying (1.1).

Throughout the paper, we use the following notations:

- $H^s(\mathbb{R}^N)$ denotes the fractional Sobolev space equipped with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^N}\left((-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v + uv\right)\mathrm{d}x, \ \|u\| = (u,u)^{\frac{1}{2}}, \text{ for any } u, v \in H^s(\mathbb{R}^N);$$

- $H_r^s(\mathbb{R}^N) = \{u \in H^s(\mathbb{R}^N): u \text{ is the radial function}\}$;
- $L^p(\mathbb{R}^N)$ (for $1 \le p \le \infty$) denotes the Lebesgue space with the norm

$$|u|_p = \left(\int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}, \ |u|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^N} |u(x)|;$$

- $D^{s,2}(\mathbb{R}^N) := \left\{u \in L^{2_s^*}(\mathbb{R}^N): \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^N), i = 1, 2, \cdots, N\right\}$

$$\langle u, v\rangle_{D^{s,2}(\mathbb{R}^N)} = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y,$$

$$\|u\|_{D^{s,2}(\mathbb{R}^N)}^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x;$$

- C denotes a positive constant and may vary in different places.

The classification of Equation (1.1) into L^2-subcritical and L^2-supercritical cases is essential for understanding its behavior. This classification is determined by the L^2-critical exponent $\bar{q} := 2 + \frac{4s}{N}$, which arises from the Gagliardo-Nirenberg inequality [4]:

$$|u|_q \le C(N,q,s)|u|_2^{1-\gamma_q} \|u\|_{D^{s,2}(\mathbb{R}^N)}^{\gamma_q}, \tag{1.3}$$

where $0 < s < 1$, $2 < q < 2_s^*$, $\gamma_q = N\left(\frac{1}{2s} - \frac{1}{qs}\right)$, $C(N,q,s)$ is a constant, $u \in H^s(\mathbb{R}^N)$. In recent years, the study of normalized solutions has become a research hot spot, such as in the whole space $\mathbb{R}^N$. Normalized solutions of the following Schrödinger equation

$$-\Delta u - g(u) = \lambda u, \ x \in \mathbb{R}^N, \tag{1.4}$$

were studied firstly in [5]-[7] considered normalized solutions of scalar equations, and [8] [9] considered normalized solutions of equations or systems in bounded domains. When $s = 1$, some authors have considered Problem (1.2) for the general case $2 < q < p \le 2_s^*$. Readers interested in this type of equation can read [10]-[12] and their references. However, there is little literature concerned about the fractional equation combined with general nonlinearities. To fill this blank, this paper will try to investigate this kind of problem.

The energy functional associated with (1.1) and the constraint are given by

$$I_q(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q \mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x, \tag{1.5}$$

and

$$S_a = \left\{u \in H^s(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a\right\}. \tag{1.6}$$

It is standard to check that $I_q \in C^1$ and a critical point of I_q constrained to S_a gives rise to a solution to (1.1), satisfying (1.6). Such a solution is usually called a normalized solution of (1.1). In this method, the parameter $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier, which depends on the solution and is not given in advance.

We define a ground state of Equation (1.1) on S_a as a solution that possesses the lowest energy among all solutions within S_a. For example, if u is a ground state of Equation (1.1) on S_a, we can get that

$$I_q(u) = \inf\left\{I_q(v) : dI_q|_{S_a}(v) = 0, v \in S_a\right\}.$$

Since the functional I_q is unbounded from below on S_a, consequently, we introduce the manifold:

$$\mathcal{P}_q(a) = \left\{u \in S_a : P_q(u) = 0\right\},$$

where

$$P_q(u)=s\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x-s\gamma_q\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x-\frac{N}{2}\int_{\mathbb{R}^N}\tilde{G}(u)\,\mathrm{d}x,$$

It is a well-known fact that any critical point of $I_q|_{S_a}$ is an element of $\mathcal{P}_q(a)$, which follows from the Pohožaev identity. Furthermore, we consider the minimizing problem:

$$c_q(a)=\inf_{u\in\mathcal{P}_q(a)}I_q(u).$$

Now, let us present the main result of this paper.

Theorem 1.1. *Assume that* $N\geq 2$, $\frac{1}{2}<s<1$, $2+\frac{4s}{N}<q<2_s^*$ *and* $(G1)$ - $(G4)$ *hold. Then for any* $a>0$ *and* $\beta>0$ *satisfying* $2_s^*C(N,\overline{q},s)\beta a^{\frac{2s}{N}}<1$, *Equation* (1.1) *has a ground state solution* $(\tilde{u}_a,\tilde{\lambda}_a)\in H_r^s(\mathbb{R}^N)\times\mathbb{R}^+$ *at positive energy level. Moreover,* $\tilde{u}_a$ *is positive and radially non-increasing and* $\tilde{\lambda}_a>0$.

The structure of the paper is outlined as follows: Section 2 presents the necessary preliminaries. In the third section, some existing conclusions as well as the proof of Theorem 1.1 are presented.

2. Preliminaries

We denote by $\mathcal{S}$ the optimal Sobolev embedding constant (see [13]), *i.e.*,

$$\mathcal{S}|u|_{2_s^*}^2\leq\|u\|_{D^{s,2}(\mathbb{R}^N)}^2,\ \text{for any } u\in H^s(\mathbb{R}^N),\ N>2s. \tag{2.1}$$

Firstly, we consider the boundedness of $G(u)$.

Lemma 2.1. *Assume that* $N>2s$, $0<s<1$, $a>0$, $2+\frac{4s}{N}<q<2_s^*$ *and* $(G1)$ - $(G4)$ *hold. Then for any* $\tau'>0$, *there exists* $C_{\tau'}>0$ *such that*

$$G(\tau)\geq C_{\tau'}|\tau|^{2+\frac{4s}{N}},\ \text{for } \tau>\tau'.$$

Proof We split the proof into three claims.

Claim 1. $G(\tau)>0$ for any $\tau\neq 0$.

By $(G1)$ - $(G3)$, the even function

$$H(\tau):=\begin{cases}\dfrac{g(\tau)\tau-2G(\tau)}{|\tau|^{2+\frac{4s}{N}}}, & \tau\neq 0,\\[2ex] \dfrac{2s}{N+2s}\beta, & \tau=0.\end{cases}$$

is continuous and increasing on $[0,\infty)$. Hence, it is clear that

$$g(\tau)\tau\geq 2G(\tau),\ \text{on } (0,\infty). \tag{2.2}$$

Noting that H is an even function by $(G1)$, then by (2.2) and $(G4)$, we have $G(\tau)>0$ for any $\tau\neq 0$.

Claim 2. $g(\tau)\tau\neq 2G(\tau)$ for any $\tau\neq 0$.

Let us assume, by contradiction, that $g(\tau_0)\tau_0=2G(\tau_0)$ for some τ_0. But since H is increasing, it is clear that $g(\tau)\tau=2G(\tau)$ on $(0,\tau_0)$. By Claim 1, we derive

$$G(\tau)=C\tau^{2},\ \text{on } (0,\tau_0),\ \text{for some } C\in\mathbb{R}^{+}. \tag{2.3}$$

However, by $(G2)$, we can deduce that there exists a τ_1 such that

$$G(\tau)<(\beta+1)|\tau|^{2+\frac{4s}{N}},\ \text{on } (0,\tau_1),$$

which contradicts (2.3). Hence, we obtain Claim 2.

Claim 3. For any $\tau'>0$, there exists $C_{\tau'}>0$ such that

$$G(\tau)\geq C_{\tau'}|\tau|^{2+\frac{4s}{N}},\ \text{for } \tau>\tau'.$$

By (2.2) and Claim 2, it is clear that $g(\tau)\tau>2G(\tau)$ on $(0,\infty)$. By (G3), we obtain

$$\frac{g(\tau)\tau-2G(\tau)}{|\tau|^{2+\frac{4s}{N}}}\geq\frac{g(\tau')\tau'-2G(\tau')}{|\tau'|^{2+\frac{4s}{N}}},\ \text{for any } \tau>\tau'. \tag{2.4}$$

By (G4) and (2.4), we have

$$\left(2_s^{*}-2\right)G(\tau)>\frac{g(\tau')\tau'-2G(\tau')}{|\tau'|^{2+\frac{4s}{N}}}|\tau|^{2+\frac{4s}{N}},\ \text{for any } \tau>\tau'.$$

Hence,

$$G(\tau)\geq C_{\tau'}|\tau|^{2+\frac{4s}{N}},\ \text{for } \tau>\tau',$$

where $C_{\tau'}=\dfrac{g(\tau')\tau'-2G(\tau')}{\left(2_s^{*}-2\right)|\tau'|^{2+\frac{4s}{N}}}>0.$ This completes the proof. □

For convenience, we put

$$\mathcal{H}(u,t)=t^{\frac{N}{2}}u(tx),\ \text{for any } x\in\mathbb{R}^{N}. \tag{2.5}$$

Next, we discuss the limits of $\left|(-\Delta)^{\frac{s}{2}}\mathcal{H}(u,t)\right|_2$ and $I_q(\mathcal{H}(u,t))$.

Lemma 2.2. *Assume that* $N>2s$, $0<s<1$, $a>0$, $2+\frac{4s}{N}<q<2_s^{*}$ *and* $(G1)$-$(G4)$ *hold. Then for any fixed* $u\in S_a$ *we have:*

1) $\left|(-\Delta)^{\frac{s}{2}}\mathcal{H}(u,t)\right|_2\to 0$, and $I_q(\mathcal{H}(u,t))\to 0$ as $t\to 0^{+}$;

2) $\left|(-\Delta)^{\frac{s}{2}}\mathcal{H}(u,t)\right|_2\to +\infty$, and $I_q(\mathcal{H}(u,t))\to -\infty$ as $t\to +\infty$.

Proof After performing a straightforward calculation, it can be shown that the following relations hold:

$$\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}\mathcal{H}(u,t)\right|^{2}\mathrm{d}x=t^{2s}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^{2}\mathrm{d}x,$$

$$\int_{\mathbb{R}^N}|\mathcal{H}(u,t)|^{2}\mathrm{d}x=a, \tag{2.6}$$

$$\int_{\mathbb{R}^N}|\mathcal{H}(u,t)|^{q}\mathrm{d}x=t^{sq\gamma_q}\int_{\mathbb{R}^N}|u|^{q}\mathrm{d}x,\ \text{for any } q\geq 2.$$

From (2.6), fixing $q > 2$, we have

$$\left|(-\Delta)^{\frac{s}{2}}\mathcal{H}(u,t)\right|_2 \to 0^+,\ \left|\mathcal{H}(u,t)\right|_q \to 0,\ \text{as } t \to 0^+,$$

and

$$\left|(-\Delta)^{\frac{s}{2}}\mathcal{H}(u,t)\right|_2 \to +\infty,\ \left|\mathcal{H}(u,t)\right|_q \to \infty,\ \text{as } t \to +\infty.$$

Employing the condition $(G2)$, we can choose a suitable constant $C_\delta > 0$ for any given $\delta > 0$, such that

$$G(u) \le (\delta + \beta)|u|^{\bar{q}} + C_\delta |u|^{2_s^*},\ \text{for any } u \in \mathbb{R}. \tag{2.7}$$

By utilizing Claim 1 of Lemma 2.1 and inequality (2.7), we can deduce the following inequalities:

$$\begin{aligned}\frac{t^{2s}}{2}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2 &\ge I_q(\mathcal{H}(u,t)) \\ &\ge \frac{t^{2s}}{2}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2 - \frac{1}{q}t^{sq\gamma_q}|u|_q^q - (\delta+\beta)t^{2s}|u|_{\bar{q}}^{\bar{q}} - C_\delta t^{2_s^* s}|u|_{2_s^*}^{2_s^*},\end{aligned} \tag{2.8}$$

and

$$I_q(\mathcal{H}(u,t)) \le \frac{t^{2s}}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x - \frac{1}{q}t^{sq\gamma_q}\int_{\mathbb{R}^N}|u|^q \mathrm{d}x. \tag{2.9}$$

Consequently, by $q > 2 + \frac{4s}{N}$, we can infer

$$I_q(\mathcal{H}(u,t)) \to 0\ \text{ as } t \to 0^+,$$

and

$$I_q(\mathcal{H}(u,t)) \to -\infty\ \text{ as } t \to +\infty.$$

This completes the proof. □

Lemma 2.3. *Assume that* $N > 2s$, $0 < s < 1$, $a > 0$, $2 + \frac{4s}{N} < q < 2_s^*$, $2_s^* C(N,\bar{q},s)\beta a^{\frac{2s}{N}} < 1$ *and* $(G1)$ - $(G4)$ *hold. Then for any* $u \in S_a$, *there exists a unique* $t_u > 0$ *such that* $\mathcal{H}(u,t_u) \in \mathcal{P}_q(a)$. *Moreover,* $I_q(\mathcal{H}(u,t_u)) > I_q(\mathcal{H}(u,t))$, *for any* $t > 0$ *with* $t \ne t_u$.

Proof Let $u \in S_a$, and $\mathcal{H}(u,t) = t^{\frac{N}{2}}u(tx)$. Then

$$I_q(\mathcal{H}(u,t)) = \frac{t^{2s}}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x - \frac{1}{q}t^{sq\gamma_q}\int_{\mathbb{R}^N}|u|^q \mathrm{d}x - \int_{\mathbb{R}^N} G(\mathcal{H}(u,t))\mathrm{d}x, \tag{2.10}$$

$$\begin{aligned}P_q(\mathcal{H}(u,t)) &= st^{2s}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x - s\gamma_q t^{sq\gamma_q}\int_{\mathbb{R}^N}|u|^q \mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N}\tilde{G}(\mathcal{H}(u,t))\mathrm{d}x \\ &= t^{2s}\left(s\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2 - s\gamma_q t^{s(q\gamma_q-2)}|u|_q^q - \frac{N}{2}\int_{\mathbb{R}^N}\frac{\tilde{G}\left(t^{\frac{N}{2}}u\right)}{\left|t^{\frac{N}{2}}u\right|^{2+\frac{4s}{N}}}|u|^{2+\frac{4s}{N}}\mathrm{d}x\right).\end{aligned} \tag{2.11}$$

It is evident that $I_q(\mathcal{H}(u,t))$ is of class C^1 and satisfies:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} I_q(\mathcal{H}(u,t)) &= st^{2s-1}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x - s\gamma_q t^{sq\gamma_q-1}\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x \\ &\quad -\frac{N}{2}t^{-1}\int_{\mathbb{R}^N}\tilde{G}(\mathcal{H}(u,t))\mathrm{d}x \\ &= \frac{1}{t}P_q(\mathcal{H}(u,t)).\end{aligned} \tag{2.12}$$

By $(G4)$ and (2.7)

$$\begin{aligned}&\frac{N}{2}\int_{\mathbb{R}^N}\tilde{G}(\mathcal{H}(u,t))\mathrm{d}x < 2_s^* s\int_{\mathbb{R}^N}G(\mathcal{H}(u,t))\mathrm{d}x \\ &\le 2_s^* s\left((\delta+\beta)t^{2s}\int_{\mathbb{R}^N}|u|^{\bar{q}}\,\mathrm{d}x + C_\delta t^{2_s^* s}\int_{\mathbb{R}^N}|u|^{2_s^*}\,\mathrm{d}x\right) \\ &\le 2_s^* s\left(C(N,\bar{q},s)a^{\frac{2s}{N}}(\delta+\beta)t^{2s}\|u\|_{D^{s,2}(\mathbb{R}^N)}^2 + Ct^{2_s^* s}\|u\|_{D^{s,2}(\mathbb{R}^N)}^{2_s^*}\right).\end{aligned}$$

It is evident that by choosing δ to be sufficiently small, we can ensure that

$$2_s^* C(N,\bar{q},s)a^{\frac{2s}{N}}(\delta+\beta) < 1. \tag{2.13}$$

Combining Equations (2.12) and (2.13), we can observe that $\frac{\mathrm{d}}{\mathrm{d}t}I_q(\mathcal{H}(u,t))(t) > 0$ for t small enough. Therefore, there exists $t_0 > 0$ such that $I_q(\mathcal{H}(u,t))(t)$ increases for $t \in (0,t_0)$. Moreover, using (2.9), we have $\lim_{t\to+\infty} I_q(\mathcal{H}(u,t)) = -\infty$. Thus, We can conclude that there exists $t_{u_1} \in \mathbb{R}^+$ such that $\mathcal{H}(u,t_{u_1}) \in \mathcal{P}_q(a)$. Now, let us suppose that there exists an alternative t_{u_2} such that $\mathcal{H}(u,t_{u_2})$ belongs to $\mathcal{P}_q(a)$. This implies that $P_q(\mathcal{H}(u,t_{u_1})) = P_q(\mathcal{H}(u,t_{u_2})) = 0$. By (2.11), we have:

$$s\gamma_q\left(t_{u_1}^{s(q\gamma_q-2)} - t_{u_2}^{s(q\gamma_q-2)}\right)\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x = \frac{N}{2}\int_{\mathbb{R}^N}\left(\frac{\tilde{G}\left(t_{u_2}^{\frac{N}{2}}u\right)}{\left|t_{u_2}^{\frac{N}{2}}u\right|^{2+\frac{4s}{N}}} - \frac{\tilde{G}\left(t_{u_1}^{\frac{N}{2}}u\right)}{\left|t_{u_1}^{\frac{N}{2}}u\right|^{2+\frac{4s}{N}}}\right)|u|^{2+\frac{4s}{N}}\,\mathrm{d}x.$$

However, this contradicts condition $(G3)$. Hence, we can conclude that $t_{u_1} = t_{u_2}$. Moreover, we also have $I_q(\mathcal{H}(u,t_u)) > I_q(\mathcal{H}(u,t))$ for all $t > 0$ with $t \neq t_u$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume that* $N > 2s$, $0 < s < 1$, $a > 0$, $2+\frac{4s}{N} < q < 2_s^*$, $2_s^* C(N,\bar{q},s)\beta a^{\frac{2s}{N}} < 1$ *and* $(G1)$ - $(G4)$ *hold. Then the following statements hold*:

1) *There exists a positive constant* ρ *such that* $\inf_{\mathcal{P}_q(a)}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2 > \rho$.

2) $c_q(a) = \inf_{\mathcal{P}_q(a)} I_q(u) > 0$.

Proof (1) For any $u \in \mathcal{P}_q(a)$, using $(G4)$, (2.7), the Gagliardo-Nirenberg inequality (1.3) and the Sobolev inequality (2.1), we obtain:

$$\begin{aligned} s\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2 &= s\gamma_q\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x+\frac{N}{2}\int_{\mathbb{R}^N}g(u)u-2G(u)\,\mathrm{d}x \\ &\le s\gamma_q C(N,q,s)a^{\frac{q(1-\gamma_q)}{2}}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{q\gamma_q}+2_s^* s\int_{\mathbb{R}^N}G(u)\,\mathrm{d}x \\ &\le s\gamma_q C(N,q,s)a^{\frac{q(1-\gamma_q)}{2}}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{q\gamma_q} \\ &\quad+2_s^* s\left(C(N,\bar{q},s)a^{\frac{2s}{N}}(\delta+\beta)\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2+C_\delta\mathcal{S}^{-\frac{2_s^*}{2}}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{2_s^*}\right). \end{aligned}$$

It is evident that by choosing δ to be sufficiently small, we can ensure that

$$2_s^* C(N,\bar{q},s)a^{\frac{2s}{N}}(\delta+\beta)<1.$$

Consequently, we can deduce that there exists $\rho>0$ such that

$$\inf_{\mathcal{P}_q(a)}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2>\rho.$$

(2) For any $t>0$ and $u\in\mathcal{P}_q(a)$, by the Gagliardo-Nirenberg inequality (1.3), (2.8) and Lemma 2.3, we have:

$$\begin{aligned} I_q(u) &\ge I_q(\mathcal{H}(u,t)) \\ &\ge \frac{t^{2s}}{2}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2-\left(\frac{1}{q}t^{sq\gamma_q}|u|_q^q+(\delta+\beta)t^{2s}|u|_{\bar{q}}^{\bar{q}}+C_\delta t^{2_s^* s}|u|_{2_s^*}^{2_s^*}\right) \\ &\ge \frac{t^{2s}}{2}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2-C_1t^{sq\gamma_q}-C_2t^{2s}-C_3t^{2_s^* s}, \end{aligned}$$

where

$$C_1=\frac{\gamma_q}{q}C(N,q,s)a^{\frac{q(1-\gamma_q)}{2}}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{q\gamma_q},$$

$$C_2=(\delta+\beta)C(N,\bar{q},s)a^{\frac{2s}{N}}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2,$$

$$C_3=C_\delta\mathcal{S}^{-\frac{2_s^*}{2}}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{2_s^*}.$$

It is evident that by choosing δ to be sufficiently small, we can ensure that

$$2C(N,\bar{q},s)a^{\frac{2s}{N}}(\delta+\beta)<1,$$

so by selecting $t=\dfrac{\sigma}{\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{\frac{1}{s}}}$ with $\sigma>0$ small enough, we deduce

$$I_q(u)>0.$$

This completes the proof. □

Combining the above lemmas, we can draw a conclusion regarding the monotonicity of $c_q(a)$, which is stated as follows:

Lemma 2.5. *Assume that* $N > 2s$, $0 < s < 1$, $a > 0$, $2 + \frac{4s}{N} < q < 2_s^*$, $2_s^* C(N, \bar{q}, s)\beta a^{\frac{2s}{N}} < 1$ *and* $(G1)$ -$(G4)$ *hold. Then the function* $a \mapsto c_q(a)$ *is nonincreasing on* $(0, \infty)$. *In particular, if* $c_q(a)$ *is achieved, then* $c_q(a) > c_q(\tilde{a})$ *for any* $\tilde{a} > a$.

Proof For any $a_2 > a_1 > 0$, there exists a sequence $\{u_n\} \subset \mathcal{P}_q(a_1)$ such that

$$I_q(u_n) < c_q(a_1) + \frac{1}{n}.$$

Let $\xi := \sqrt{\frac{a_2}{a_1}} \in (1, \infty)$ and define

$$v_n(x) := \xi^{\frac{2s-N}{2s}} u_n\left(\xi^{-\frac{1}{s}} x\right).$$

We have $|v_n|_2^2 = a_2$, $\left|(-\Delta)^{\frac{s}{2}} v_n\right|_2 = \left|(-\Delta)^{\frac{s}{2}} u_n\right|_2$, and $\left|v_n\right|_q = \xi^{1-\gamma_q}\left|u_n\right|_q$.

For any $u \in \mathbb{R} \setminus \{0\}$, let us define

$$B_u(\sigma) := G(u) - \sigma^{\frac{N}{s}} G\left(\sigma^{\frac{2s-N}{2s}} u\right), \text{ for any } \sigma \geq 1.$$

Clearly, $B_u(1) = 0$. By $(G4)$, we get

$$B_u'(\sigma) = -\frac{N}{s}\sigma^{\frac{N}{s}-1}\left[G\left(\sigma^{\frac{2s-N}{2s}} u\right) - \frac{N-2s}{2N} g\left(\sigma^{\frac{2s-N}{2s}} u\right)\sigma^{\frac{2s-N}{2s}} u\right] < 0.$$

Then, we deduce that $B_u(\xi) < 0$, which implies that

$$\int_{\mathbb{R}^N} G(\mathcal{H}(u_n, t_n))\mathrm{d}x < \xi^{\frac{N}{s}} \int_{\mathbb{R}^N} G\left(\xi^{\frac{2s-N}{2s}} \mathcal{H}(u_n, t_n)\right)\mathrm{d}x. \tag{2.14}$$

Using Lemma 2.3, we can find $t_n > 0$ such that $\{\mathcal{H}(v_n, t_n)\} \subset \mathcal{P}_q(a_2)$. By applying (2.10), (2.14) and Lemma 2.4, we obtain

$$\begin{aligned} c_q(a_2) &\leq I_q(\mathcal{H}(v_n, t_n)) \\ &= I_q(\mathcal{H}(u_n, t_n)) - \frac{1}{q} t_n^{sq\gamma q} \int_{\mathbb{R}^N} |v_n|^q \mathrm{d}x + \frac{1}{q} t_n^{sq\gamma q} \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \left[G(\mathcal{H}(u_n, t_n)) - G(\mathcal{H}(v_n, t_n))\right]\mathrm{d}x \\ &\leq I_q(\mathcal{H}(u_n, t_n)) + \int_{\mathbb{R}^N} \left[G(\mathcal{H}(u_n, t_n)) - \xi^{\frac{N}{s}} G\left(\xi^{\frac{2s-N}{2s}} \mathcal{H}(u_n, t_n)\right)\right]\mathrm{d}x \\ &< I_q(\mathcal{H}(u_n, t_n)) \leq I_q(u_n) < c_q(a_1) + \frac{1}{n}, \end{aligned}$$

which implies $c_q(a_2) \leq c_q(a_1)$ by taking the limit as $n \to \infty$.

Next, we assume that $c_q(a)$ is achieved, meaning that there exists $\tilde{u} \in \mathcal{P}_q(a)$ such that $I_q(\tilde{u}) = c_q(a)$. Let us consider $a' > a$, and define $\tilde{\xi} = \sqrt{\frac{a'}{a}} \in (1, \infty)$

and $\tilde{v}(x) := \tilde{\xi}^{\frac{2s-N}{2s}} \tilde{u}\left(\tilde{\xi}^{-\frac{1}{s}} x\right)$. It can be observed that $|\tilde{u}|_2^2 = a'$, $\left|(-\Delta)^{\frac{s}{2}} \tilde{v}\right|_2 = \left|(-\Delta)^{\frac{s}{2}} \tilde{u}\right|_2$, and $|\tilde{v}|_q = \tilde{\xi}^{1-\gamma_q} |\tilde{u}|_q$. By Lemma 2.3, there exists $t_0 > 0$ such that $\mathcal{H}(\tilde{v}, t_0) \in \mathcal{P}_q(a')$.

Consequently, using (2.10), (2.14), and Lemma 2.4, we have

$$
\begin{aligned}
c_q(a') &\le I_q\left(\mathcal{H}(\tilde{v}, t_0)\right) \\
&= I_q\left(\mathcal{H}(\tilde{u}, t_0)\right) - \frac{1}{q} t_0^{sq\gamma_q} \int_{\mathbb{R}^N} |\tilde{v}|^q \,\mathrm{d}x + \frac{1}{q} t_0^{sq\gamma_q} \int_{\mathbb{R}^N} |\tilde{u}|^q \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left[G\left(\mathcal{H}(\tilde{u}, t_0)\right) - G\left(\mathcal{H}(\tilde{v}, t_0)\right) \right] \mathrm{d}x \\
&\le I_q\left(H(\tilde{u}, t_0)\right) + \int_{\mathbb{R}^N} \left[G\left(\mathcal{H}(\tilde{u}, t_0)\right) - \tilde{\xi}^{\frac{N}{s}} G\left(\tilde{\xi}^{\frac{2s-N}{2s}} \mathcal{H}(\tilde{u}, t_0) \right) \right] \mathrm{d}x \\
&< I_q\left(\mathcal{H}(\tilde{u}, t_0)\right) \le I_q(\tilde{u}) = c_q(a),
\end{aligned}
$$

which shows that $c_q(a') < c_q(a)$. This completes the proof. □

The following lemma is crucial for proving the achievability of $c_{(}a)$.

Lemma 2.6. *Assume that* $N \ge 2$, $0 < s < 1$, $a > 0$, $2 + \frac{4s}{N} < q < 2_s^*$, $2_s^* C(N, \bar{q}, s) \beta a^{\frac{2s}{N}} < 1$ *and* $(G1)$ - $(G4)$ *hold. Let* $\{u_n\} \subset \mathcal{P}_q(a)$ *be the minimizing sequence of* $c_q(a)$, *then* $\{u_n\}$ *is bounded in* $H^s(\mathbb{R}^N)$.

Proof Let us assume, by contradiction, that $t_n^{2s} := \left|(-\Delta)^{\frac{s}{2}} u_n\right|_2^2 \to \infty$. Let $w_n = \mathcal{H}\left(u_n, \frac{1}{t_n}\right)$, it can be observed that $\left|(-\Delta)^{\frac{s}{2}} w_n\right|_2^2 = 1$, which implies that the sequence $\{w_n\}$ is bounded in the Sobolev space $H^s(\mathbb{R}^N)$. Let

$$
v := \limsup_{n \to \infty} \left(\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |w_n|^2 \,\mathrm{d}x \right).
$$

Obviously, $v \ge 0$.

If $v = 0$, according to the Lions' Lemma [14], we deduce that

$$
\int_{\mathbb{R}^N} |w_n|^p \,\mathrm{d}x \to 0, \ 2 < p < 2_s^*. \tag{2.15}
$$

Furthermore, by $(G2)$, for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$
G(u) \le (C_\delta + \beta)|u|^{\bar{q}} + \delta |u|^{2_s^*}, \ \text{for any } u \in \mathbb{R}. \tag{2.16}
$$

Using (2.16), it can be seen that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} G(w_n) \,\mathrm{d}x \le (C_\delta + \beta) \lim_{n \to \infty} \int_{\mathbb{R}^N} |w_n|^{\bar{q}} \,\mathrm{d}x + \delta \lim_{n \to \infty} \int_{\mathbb{R}^N} |w_n|^{2_s^*} \,\mathrm{d}x.
$$

For small enough δ, we have

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} G(w_n) \,\mathrm{d}x = 0, \ \lim_{n \to \infty} \int_{\mathbb{R}^N} G\left(\mathcal{H}(w_n, t)\right) \mathrm{d}x = 0, \ \text{for any } t > 0.
$$

Since $\mathcal{H}(w_n, t_n) = u_n$, applying Lemma 2.3, for any $t > 0$ there holds

$$\begin{aligned} c_q(a)+o(1) &= I_q(u_n)=I_q(\mathcal{H}(w_n,t_n))\geq I_q(\mathcal{H}(w_n,t)) \\ &= \frac{t^{2s}}{2}\left|(-\Delta)^{\frac{s}{2}}w_n\right|_2^2-\frac{t^{sq\gamma_q}}{q}\int_{\mathbb{R}^N}|w_n|^q\,\mathrm{d}x-\int_{\mathbb{R}^N}G(\mathcal{H}(w_n,t))\mathrm{d}x \\ &= \frac{t^{2s}}{2}+o(1). \end{aligned}$$

This contradicts the fact that $c_q(a)<+\infty$ by taking sufficiently large t.

Therefore, we can conclude that $v>0$. Then, up to a subsequence, there exists $\{z_n\}\subset\mathbb{R}^N$ such that $w_n(\cdot+z_n)\rightharpoonup w\neq 0$ in $H^s(\mathbb{R}^N)$. Then, by (2.9) and Lemma 2.4, we have

$$\begin{aligned} 0<c_q(a)+o(1) &= I_q(u_n)=I_q(\mathcal{H}(w_n,t_n)) \\ &\leq \frac{1}{2}t_n^{2s}\left|(-\Delta)^{\frac{s}{2}}w_n\right|_2^2-\frac{t_n^{sq\gamma_q}}{q}\int_{\mathbb{R}^N}|w_n|^q\,\mathrm{d}x \\ &= t_n^{2s}\left(\frac{1}{2}-\frac{1}{q}t_n^{s(q\gamma_q-2)}\int_{\mathbb{R}^N}|w_n(x+z_n)|^q\,\mathrm{d}x\right) \\ &\to-\infty,\ \text{as } t_n\to\infty. \end{aligned}$$

This contradiction implies t_n is bounded, which in turn implies that the sequence $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. □

3. Existence Result and Proof of Theorem 1.1

Lemma 3.1. *Assume that* $N\geq 2$, $\frac{1}{2}<s<1$, $a>0$, $2+\frac{4s}{N}<q<2_s^*$, $2_s^*C(N,\bar{q},s)\beta a^{\frac{2s}{N}}<1$ *and* $(G1)$-$(G4)$ *hold. Then* $c_q(a)$ *is attained by a positive and radially non-increasing function.*

Proof Let $\{u_n\}\subset\mathcal{P}_q(a)$ be the minimizing sequence of $c_q(a)$, and by the assumption $(G1)$, we have $\{|u_n|\}\subset\mathcal{P}_q(a)$ and $I_q(|u_n|)\to c_q(a)$. Let $\{v_n\}=\{|u_n|^*\}$ be the Schwartz symmetrization rearrangement of $\{|u_n|\}$. Then, from [15] [16], we have

$$\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}v_n\right|^2\mathrm{d}x\leq\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}|u_n|\right|^2\mathrm{d}x,$$

$$\int_{\mathbb{R}^N}|v_n|^q\,\mathrm{d}x=\int_{\mathbb{R}^N}|u_n|^q\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^N}G(v_n)\mathrm{d}x=\int_{\mathbb{R}^N}G(|u_n|)\mathrm{d}x.$$

We can see that $\{v_n\}\subset S_{a,r}:=S_a\cap H_r^s(\mathbb{R}^N)=\{u\in S_a:u(x)=u(|x|)\}$.

Moreover, $P_q(v_n)\leq P_q(|u_n|)=0$. By Lemma 2.3, there exists $\tau_n>0$ such that $P_q(\mathcal{H}(v_n,\tau_n))=0$. We also have

$$\begin{aligned} I_q(\mathcal{H}(v_n,\tau_n)) &= \frac{\tau_n^{2s}}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}v_n\right|^2\mathrm{d}x-\frac{\tau_n^{sq\gamma_q}}{q}\int_{\mathbb{R}^N}|v_n|^q\,\mathrm{d}x-\tau_n^{-N}\int_{\mathbb{R}^N}G\left(\tau_n^{\frac{N}{2}}v_n\right)\mathrm{d}x \\ &\leq \frac{\tau_n^{2s}}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}|u_n|\right|^2\mathrm{d}x-\frac{\tau_n^{sq\gamma_q}}{q}\int_{\mathbb{R}^N}|u_n|^q\,\mathrm{d}x-\tau_n^{-N}\int_{\mathbb{R}^N}G\left(\tau_n^{\frac{N}{2}}|u_n|\right)\mathrm{d}x \\ &= I_q(\mathcal{H}(|u_n|,\tau_n))\leq I_q(|u_n|). \end{aligned}$$

Therefore, $\{\mathcal{H}(v_n,\tau_n)\}\subset\mathcal{P}_{q,r}(a):=\mathcal{P}_q(a)\cap H_r^s(\mathbb{R}^N)=\{u\in\mathcal{P}_q(a):u(x)=u(|x|)\}$ is a minimizing sequence of $c_q(a)$. By Lemma 2.6, it is evident that $\{\mathcal{H}(v_n,\tau_n)\}$ is bounded in $H_r^s(\mathbb{R}^N)$. Therefore, there exists $u\in H_r^s(\mathbb{R}^N)$ such that $\mathcal{H}(v_n,\tau_n)\rightharpoonup u$ in $H_r^s(\mathbb{R}^N)$, $\mathcal{H}(v_n,\tau_n)\to u$ in $L^t(\mathbb{R}^N)$ with $t\in(2,2_s^*)$ and $\mathcal{H}(v_n,\tau_n)\to u$, *i.e.*, in $\mathbb{R}^N$. Consequently,

$$\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x\le\liminf_{n\to\infty}\int_{\mathbb{R}^N}|H(v_n,\tau_n)|^2\,\mathrm{d}x=a, \tag{3.1}$$

$$\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\le\liminf_{n\to\infty}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}\mathcal{H}(v_n,\tau_n)\right|^2\mathrm{d}x. \tag{3.2}$$

Moreover, since $\{\mathcal{H}(v_n,\tau_n)\}\subset H_r^s(\mathbb{R}^N)$, by Strauss' inequality [17]

$$|\mathcal{H}(v_n,\tau_n)(x)|\le C_N|\mathcal{H}(v_n,\tau_n)|_2^{1-\frac{1}{2s}}\left|(-\Delta)^{\frac{s}{2}}\mathcal{H}(v_n,\tau_n)\right|_2^{\frac{1}{2s}}|x|^{\frac{1-N}{2}}\ \text{a.e. on }\mathbb{R}^N.$$

This implies $\mathcal{H}(v_n,\tau_n)\to0$ as $|x|\to\infty$. By $(G2)$, one has

$$\lim_{|u|\to0}\frac{G(u)}{u^2+u^{2_s^*}}=0\ \text{and}\ \lim_{|u|\to+\infty}\frac{G(u)}{u^2+u^{2_s^*}}=0.$$

Since $\{\mathcal{H}(v_n,\tau_n)\}$ is bounded in $H_r^s(\mathbb{R}^N)$, then, by the Sobolev inequality (2.1), we have

$$\int_{\mathbb{R}^N}|\mathcal{H}(v_n,\tau_n)|^2+|\mathcal{H}(v_n,\tau_n)|^{2_s^*}\,\mathrm{d}x\le M,\ \text{for some positive } M.$$

Then, according to [18], we have

$$\int_{\mathbb{R}^N}G(\mathcal{H}(v_n,\tau_n))\mathrm{d}x\to\int_{\mathbb{R}^N}G(u)\mathrm{d}x, \tag{3.3}$$

$$\int_{\mathbb{R}^N}g(\mathcal{H}(v_n,\tau_n))\mathcal{H}(v_n,\tau_n)\mathrm{d}x\to\int_{\mathbb{R}^N}g(u)u\mathrm{d}x. \tag{3.4}$$

Therefore, Combining (3.1)-(3.4), we obtain

$$\begin{aligned}I_q(\mathcal{H}(v_n,\tau_n))\to&\left(\frac{\gamma_q}{2}-\frac{1}{q}\right)\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x+\frac{N}{4s}\int_{\mathbb{R}^N}g(u)u\mathrm{d}x\\&-\frac{N+2s}{2s}\int_{\mathbb{R}^N}G(u)\mathrm{d}x=c_q(a).\end{aligned}$$

This implies that $u\not\equiv0$ by Lemma 2.4, and $P_q(u)\le0$ by (3.2).

Set $|u|_2^2:=a_0\le a$, by Lemma 2.3, there exists a unique $t_u\in(0,1]$ such that $\mathcal{H}(u,t_u)\in\mathcal{P}_q(a_0)$. We claim that

$$I_q(u)-I_q(\mathcal{H}(u,t_u))\ge\frac{1-t_u^{2s}}{2s}P_q(u)\ \text{for } 0<t_u\le1.$$

By direct calculation, we obtain

$$\begin{aligned}&I_q(u)-I_q(\mathcal{H}(u,t_u))\\&=\frac{1-t_u^{2s}}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\frac{t_u^{sq\gamma_q}-1}{q}\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x\\&\quad+t_u^{-N}\int_{\mathbb{R}^N}G\left(t_u^{\frac{N}{2}}u\right)\mathrm{d}x-\int_{\mathbb{R}^N}G(u)\mathrm{d}x\end{aligned}$$

$$\begin{aligned} &= \frac{1-t_u^{2s}}{2s} P_q(u) + \left(\frac{1-t_u^{2s}}{2}\gamma_q + \frac{t_u^{sq\gamma_q}-1}{q} \right) \int_{\mathbb{R}^N} |u|^q \, \mathrm{d}x + \int_{\mathbb{R}^N} A(u) \mathrm{d}x \\ &\geq \frac{1-t_u^{2s}}{2} P_q(u) + \int_{\mathbb{R}^N} A(u) \mathrm{d}x, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} A(u) &= \frac{N\left(1-t_u^{2s}\right)}{4s} g(u)u - \frac{2s+N\left(1-t_u^{2s}\right)}{2s} G(u) + t_u^{-N} G\left(t_u^{\frac{N}{2}} u \right) \\ &= \int_{t_u}^{1} \frac{N}{2} t^{2s-1} |u|^{2+\frac{4s}{N}} \left[\frac{g(u)u - 2G(u)}{|u|^{2+\frac{4s}{N}}} - \frac{g\left(t^{\frac{N}{2}} u \right) u - 2G\left(t^{\frac{N}{2}} u \right)}{\left| t^{\frac{N}{2}} u \right|^{2+\frac{4s}{N}}} \right] \mathrm{d}t \geq 0 \end{aligned} \tag{3.6}$$

The last inequality sign can be justified by $(G3)$. Consequently, combining (3.5) and (3.6), we obtain

$$I_q(u) - I_q\left(\mathcal{H}(u,t_u)\right) \geq \frac{1-t_u^{2s}}{2s} P_q(u) \text{ for } 0 < t_u \leq 1. \tag{3.7}$$

Then

$$\begin{aligned} c_q(a_0) &\leq I_q\left(\mathcal{H}(u,t_u)\right) \leq I_q(u) - \frac{1-t_u^{2s}}{2s} P_q(u) \leq I_q(u) - \frac{1}{2s} P_q(u) \\ &= \left(\frac{\gamma_q}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u|^q \, \mathrm{d}x + \frac{N}{4s} \int_{\mathbb{R}^N} \tilde{G}(u) \mathrm{d}x - \int_{\mathbb{R}^N} G(u) \mathrm{d}x \\ &= \lim_{n\to\infty} \left[\left(\frac{\gamma_q}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} \left| \mathcal{H}(v_n,\tau_n) \right|^q \mathrm{d}x \right. \\ &\quad \left. + \frac{N}{4s} \int_{\mathbb{R}^N} \tilde{G}\left(\mathcal{H}(v_n,\tau_n)\right) \mathrm{d}x - \int_{\mathbb{R}^N} G\left(\mathcal{H}(v_n,\tau_n)\right) \mathrm{d}x \right] \\ &= \lim_{n\to\infty} \left(I_q\left(\mathcal{H}(v_n,\tau_n)\right) - \frac{1}{2s} P_q\left(\mathcal{H}(v_n,\tau_n)\right) \right) \\ &= c_q(a) \leq c_q(a_0) \end{aligned}$$

This shows $c_q(a_0) = c_q(a)$ and $c_q(a_0)$ is attained. By Lemma 2.5, we obtain that $a_0 = a$, $t_u = 1$, and $I_q(u) = c_q(a)$. Thus, we can deduce that $\mathcal{H}(v_n,\tau_n) \to u$ in $H^s\left(\mathbb{R}^N\right)$ and u belongs to $\mathcal{P}_q(a)$, achieving $c_q(a)$. This completes the proof. □

Lemma 3.2. *Assume that* $N \geq 2$, $\frac{1}{2} < s < 1$, $a > 0$, $2 + \frac{4s}{N} < q < 2_s^*$, $2_s^* C\left(N,\bar{q},s\right) \beta a^{\frac{2s}{N}} < 1$, *and* (G_1) - (G_4) *hold. If* $\bar{u} \in \mathcal{P}_q(a)$, $I_q(\bar{u}) = c_q(a)$, *then* $\bar{u}$ *is a critical point of* $I_q\big|_{S_a}$.

Proof Suppose $\bar{u}$ is not a critical point of $I_q\big|_{S_a}$, then there exist positive constants δ and θ such that for any $u \in S_a$, whenever $\|u - \bar{u}\| \leq 3\delta$, it holds that $\left\| I_q'(u) \right\| \geq \theta$.

To begin with, we clarify that

$$\lim_{t\to 1}\left\|\mathcal{H}(\bar{u},t)-\bar{u}\right\|=0. \tag{3.8}$$

By (3.5), one has

$$I_q\left(\mathcal{H}(\bar{u},t)\right)=I_q(\bar{u})-h(\bar{u},t)=c_q(a)-h(\bar{u},t),\ \text{for any } t>0, \tag{3.9}$$

where

$$h(\bar{u},t)=\int_{\mathbb{R}^N}\frac{N\left(1-t^{2s}\right)}{4s}g(\bar{u})\bar{u}+\left(\frac{1-t^{2s}}{2}\gamma_q+\frac{t^{sq\gamma_q}-1}{q}\right)\int_{\mathbb{R}^N}|\bar{u}|^q\,\mathrm{d}x$$
$$-\int_{\mathbb{R}^N}\frac{2s+N\left(1-t^{2s}\right)}{2s}G(\bar{u})\,\mathrm{d}x+t^{-N}\int_{\mathbb{R}^N}G\left(t^{\frac{N}{2}}\bar{u}\right)\mathrm{d}x.$$

By (3.6), we know that $h(\bar{u},t)>0$ for $t\in(0,1)\cup(1,+\infty)$ and $\bar{u}\in H^s\left(\mathbb{R}^N\right)\setminus\{0\}$. By Lemma 2.3, there exists $T_1\in(0,1)$ and $T_2\in(1,+\infty)$ such that

$$P_q\left(\mathcal{H}(\bar{u},T_1)\right)>0,\ P_q\left(\mathcal{H}(\bar{u},T_2)\right)<0. \tag{3.10}$$

Applying Willem's quantitative deformation Lemma [19]. Let

$$S:=B(\bar{u},\delta)\cap S_a,\ \varepsilon:=\min\left\{\frac{h(T_1,\bar{u})}{4},\frac{h(T_2,\bar{u})}{4},1,\frac{\delta\theta}{8}\right\}.$$

Then there exists $\eta\in C\left([0,1]\times S_a,S_a\right)$ such that

(a) $\eta(1,u)=u$ if $I_q(u)<c_q(a)-2\varepsilon$ or $I_q(u)>c_q(a)+2\varepsilon$;

(b) $\eta\left(1,I_q^{c_q(a)+\varepsilon}\cap S\right)\subset I_q^{c_q(a)-\varepsilon}$;

(c) $I_q\left(\eta(1,u)\right)\le I_q(u)$ for any $u\in S_a$.

According to Lemma 2.3, $I_q\left(\mathcal{H}(\bar{u},t)\right)\le I_q(\bar{u})=c_q(a)$ for $t>0$, then it follows from (3.8) and (b) that

$$I_q\left(\eta\left(1,\mathcal{H}(\bar{u},t)\right)\right)\le c_q(a)-\varepsilon,\ \text{for any } t>0,\ |t-1|<\delta_1. \tag{3.11}$$

By (c) and (3.9), one has

$$I_q\left(\eta\left(1,\mathcal{H}(\bar{u},t)\right)\right)\le I_q\left(\mathcal{H}(\bar{u},t)\right)\le c_q(a)-\delta_2,\ \text{for any } t>0,\ |t-1|\ge\delta_1, \tag{3.12}$$

where

$$\delta_2:=\min\left\{h(1-\delta_1,\bar{u}),h(1+\delta_1,\bar{u})\right\}>0.$$

Combining (3.11) with (3.12), we have

$$\max_{t\in[T_1,T_2]}I_q\left(\eta\left(1,\mathcal{H}(\bar{u},t)\right)\right)<c_q(a).$$

Next, we claim that $\eta\left(1,\mathcal{H}(\bar{u},t)\right)\cap\mathcal{P}_q(a)\ne\varnothing$ for some $t\in[T_1,T_2]$.

Define $\psi_q(t):=P_q\left(\eta\left(1,\mathcal{H}(\bar{u},t)\right)\right)$ for $t>0$. It follows from (3.9) and (a) that $\eta\left(1,\mathcal{H}(\bar{u},t)\right)=\mathcal{H}(\bar{u},t)$ for $t=T_1$ and $t=T_2$, which, together with (3.10), implies

$$\psi_q(T_1)=P_q\left(\mathcal{H}(\bar{u},T_1)\right)>0,\ \psi_q(T_2)=P_q\left(\mathcal{H}(\bar{u},T_2)\right)<0.$$

Since $\psi_q(t)$ is continuous on $(0,\infty)$, then there exists $t_0 \in [T_1, T_2]$ such that $\psi_q(t_0) = 0$. Hence, $\eta(1, \mathcal{H}(\bar{u}, t_0)) \cap \mathcal{P}_q(a) \neq \varnothing$, contradicting the definition of $c_q(a)$. This completes the proof. □

Proof of Theorem 1.1 It follows from Lemmas 3.1 and 3.2 that there exists $\tilde{u}_a \in \mathcal{P}_q(a)$, such that

$$I_q(\tilde{u}_a) = c_q(a) \text{ and } I_q'\big|_{S_a}(\tilde{u}_a) = 0$$

By the Lagrange multiplier theorem, there exists a Lagrange multiplier $\tilde{\lambda}_a \in \mathbb{R}$ such that $(\tilde{u}_a, \tilde{\lambda}_a)$ solves

$$(-\Delta)^s u + \lambda u = g(u) + |u|^{q-2} u.$$

Then, we have

$$\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} \tilde{u}_a\right|^2 \mathrm{d}x + \tilde{\lambda}_a \int_{\mathbb{R}^N} |\tilde{u}_a|^2 \mathrm{d}x - \int_{\mathbb{R}^N} g(\tilde{u}_a)\tilde{u}_a \mathrm{d}x - \int_{\mathbb{R}^N} |\tilde{u}_a|^q \mathrm{d}x = 0. \quad (3.13)$$

Using the fact that $\tilde{u}_a \in \mathcal{P}_q(a)$, we have

$$s\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} \tilde{u}_a\right|^2 \mathrm{d}x - s\gamma_q \int_{\mathbb{R}^N} |\tilde{u}_a|^q \mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} g(\tilde{u}_a)\tilde{u}_a - 2G(\tilde{u}_a)\mathrm{d}x = 0. \quad (3.14)$$

Combining (3.13) with (3.14), we obtain

$$\begin{aligned}\tilde{\lambda}_a &= \frac{1}{a}\left(\int_{\mathbb{R}^N} g(\tilde{u}_a)\tilde{u}_a \mathrm{d}x + \int_{\mathbb{R}^N} |\tilde{u}_a|^q \mathrm{d}x - \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} \tilde{u}_a\right|^2 \mathrm{d}x\right) \\ &= \frac{1}{a}\left[(1-\gamma_q)\int_{\mathbb{R}^N} |\tilde{u}_a|^q \mathrm{d}x + \frac{2s-N}{2s}\int_{\mathbb{R}^N} g(\tilde{u}_a)\tilde{u}_a \mathrm{d}x + \frac{N}{s}\int_{\mathbb{R}^N} G(\tilde{u}_a)\mathrm{d}x\right].\end{aligned}$$

By (G4), it implies $\tilde{\lambda}_a > 0$. Thus $(\tilde{u}_a, \tilde{\lambda}_a) \in H_r^s(\mathbb{R}^N) \times \mathbb{R}^+$ is a normalized ground state solution of Equation (1.1). We assert that $\tilde{u}_a > 0$ by the strong maximum principle. Furthermore, by Lemma 3.1, $\tilde{u}_a$ is radially non-increasing.

Conflicts of Interest

The author declares no conflicts of interest.

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